PMM U.S.S.R., Vol.53, No.3, pp.350-355, 1989
Printed in Great Britain
0021-8928/89 $10.00+0.00

THE FORCING OUT OF SALINE WATER BY FRESH WATER DURING FILTRATION FROM A MOLE IRRIGATOR*

E.N. BERESLAVSKII

The question of the possible dimensions of the lens of fresh waters which is formed during filtration from a mole irrigator is investigated using Polubarinova-Kochina's method /1/ based on the analytical theory of differential equations. The problem reduces to an investigation of an ordinary second-order linear differential equation of the Fuchs class where fundamental difficulties arise during its integration. The difficulties are primarily due to the fact that this equation belongs to a class which has not been thoroughly studied and, also, due to the fact that the coefficients of the equation contain unknown parameters, the determination of which is one of the basic and most difficult problems of the whole theory.

A method is proposed for setting up the integrals and for determining the unknown constants for one class of Fuchs equations with four singular points. It is shown that the Fourier method /1, 2/ also yields similar results in the case under consideration. Results of numerical calculations are presented together with an analysis of the effect of the decisive physical parameters of the scheme on the filtration characteristics.

A fairly complete bibliography of papers dealing with the study of different mathematical models of irrigation within the soil is contained in the review /3/.

1. *Formulation of the problem.* Planar steady state filtration from a mole irrigator takes place in the lens of fresh waters which is formed in a homogeneous and isotropic layer of soil over stationary saline ground waters. The entrance of water into the lens is compensated by its evaporation from the free surface at a constant intensity ε $(\neq 0)$ which is related to the filtration coefficient of the ground. In the initial treatment we shall replace the irrigator by a point source located at a point A. In view of the symmetry of the problem, it is sufficient to confine ourselves to the right half of the domain of the motion $z = x + iy$ which is depicted in Fig.1.

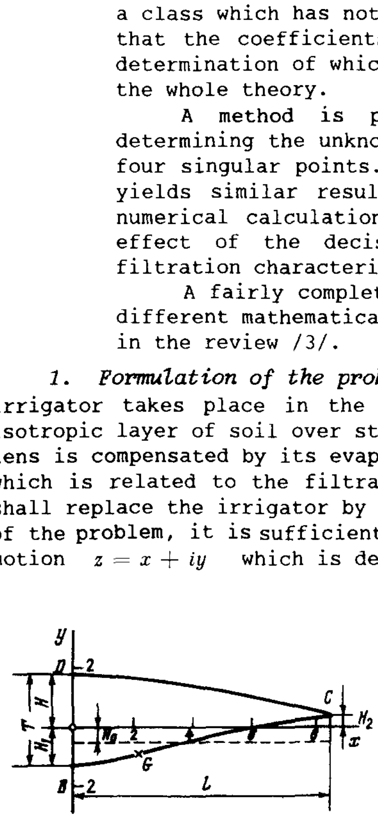

Fig.1

The problem involves determining the depression curve CD and the line of separation BC subject to the following boundary conditions:

$$AB\colon x = 0,\quad \psi = 0;\ BC\colon \varphi - \rho y = H + (1+\rho) H_2,\quad \psi = 0 \tag{1.1}$$
$$CD\colon \varphi + y = H,\quad \psi + \varepsilon x = {}^1/_2 Q;\ AD\colon x = 0,\quad \psi = {}^1/_2 Q$$
$$\rho = \rho_2/\rho_1 - 1,\quad \rho > \varepsilon$$

Here φ and ψ are mutually selfadjoint harmonic functions within the domain z and are velocity potential and stream function referred to the coefficient of filtration of the ground, ρ_1 and ρ_2 are the densities of the fresh and saline waters and Q is the required filtration flow rate per unit length of the irrigator referred to the filtration coefficient.

By putting $x = L$ in the second condition for the part CD, we get

$$Q = 2\varepsilon L \tag{1.2}$$

This relationship expresses the equality of the flow rate from the irrigator and the amount of evaporation from the free surface under steady-state filtration conditions.

The auxiliary variable $\xi = \zeta + i\eta$ and the following functions are introduced: $z(\xi)$, which conformally maps the upper half plane onto the domain z (the correspondence of the points is indicated in Fig.2,a), the complex velocity $w = d\omega/dz$ and, also,

*Prikl. Matem. Mekhan., 53, 3, 455-461, 1989

$$F(\xi) = d\omega/d\xi, \quad Z(\xi) = dz/d\xi \tag{1.3}$$

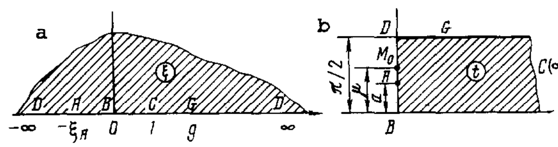

Fig.2

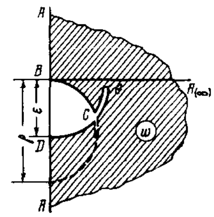

Fig.3

2. *Construction of the function w.* The domain, over which the complex velocity changes, corresponding to the boundary conditions (1.1) is shown in Fig.3. This domain is a curved quadrilateral $BCGD$ with right angles at the vertices B and D, an angle $\pi\nu = \arccos [(2\varepsilon + \rho\varepsilon - \rho)/(\rho(1+\varepsilon))]$ at the vertex C so that $z_C = [\sqrt{\varepsilon(\rho+1)(\rho-\varepsilon)} - i\varepsilon]/(\rho + 1 - \varepsilon)$ and an angle of 2π at the vertex of the cut G.

It is well-known /1, 4/ that the problem of the conformal mapping of a curved quadrilateral onto the upper half plane is intimately associated with a certain linear second-order differential equation of the Fuchs class with four singular points. In the case being considered, this equation has the form

$$v'' + \left(\frac{1}{2\xi} + \frac{1-\nu}{\xi-1} - \frac{1}{\xi-g}\right)v' + \frac{\nu(1+\nu)\xi - \lambda}{4\xi(\xi-1)(\xi-g)}v = 0 \tag{2.1}$$

It is also known /1, 4/ that difficulties of a fundamental nature arise during the integration of equations of this type, associated with the fact that, apart from the undetermined quantity g, the coefficients of Eqs.(2.1) also contain the so-called accessor parameter λ regarding which there is also no a priori knowledge. These parameters, which cannot be completely determined from a specified quadrilateral, have to be found at the same time as the integrals are constructed and, up to the present time, there is no sufficiently general and convenient method for determining them. In order to study the properties of the integrals it is therefore necessary to use different indirect methods. Here, papers /5, 6/ should first be noted in which the problem of linear conjunction was used to construct certain solutions.

We shall show that, in the case under consideration, solutions of Eq.(2.1) can be found directly and, moreover, in terms of elementary functions. Let us make the change of variables

$$\xi = \operatorname{th}^2 t \tag{2.2}$$

in Eq.(2.1) which transforms the upper half plane ξ into the half band $u > 0$, $0 < v < \pi/2$ of the plane $t = u + iv$ (Fig.2,b) and seek the solution of the resulting equation in the form

$$v_1 = (C_1 \operatorname{ch} t \operatorname{ch} \nu t + C_2 \operatorname{sh} t \operatorname{sh} \nu t)/\operatorname{ch}^{1+\nu} t \tag{2.3}$$

where C_1 and C_2 are certain arbitrary constants which do not vanish simultaneously. It can be shown that the result of the substitution of (2.3) into the transformed Eq.(2.1) vanishes identically if the following two conditions are satisfied:

$$g\nu[(\nu-1)C_1 + 2C_2] - \lambda C_1 = 0 \tag{2.4}$$
$$g(2 + \nu - \nu^2)C_2 - \lambda C_2 = 2(C_2 - \nu C_1)$$

The required parameters g and λ are found from this system.

A second linearly independent solution is constructed in an analogous manner

$$v_2 = (C_1 \operatorname{ch} t \operatorname{sh} \nu t + C_2 \operatorname{sh} t \operatorname{ch} \nu t)/\operatorname{ch}^{1+\nu} t \tag{2.5}$$

A function which conformally maps the half band of the plane t onto the curved quadrilateral of the plane w must be expressed in terms of a ratio of linear combinations of v_1 and v_2.

If such combinations are made up and use is made of the matching of the points B, C and D in the t and z planes, we get

$$w = \gamma\rho f_1(t)/f_2(t),\quad f_1(t) = \operatorname{ch} t \operatorname{sh} \nu t + C \operatorname{sh} t \operatorname{ch} \nu t,\quad f_2(t) = \operatorname{ch} t (\operatorname{ch} \nu t + i\gamma \operatorname{sh} \nu t) + C \operatorname{sh} t (\operatorname{sh} \nu t + i\gamma \operatorname{ch} \nu t) \tag{2.6}$$

$$\gamma = \sqrt{\varepsilon/[(\rho+1)(\rho-\varepsilon)]},\quad C = \operatorname{ctg} a\,(1 - \gamma \operatorname{tg} \nu a)/(\gamma + \operatorname{tg} \nu a)$$

An analysis of expression (2.6) shows that the function w attains its extremum on the side CD of the quadrilateral when $\lambda^{-1} < C < -1$ or $0 < C < 1$ which corresponds to $g > 1$ (the cut shown in Fig.3 corresponds to this case). When $-1 < C < -\lambda$ or $C > 1$, the parameter $g < 1$ and the extremum of the function w is attained on the side BC (the broken line cut in Fig.3 corresponds to this case).

We note that, if Eqs.(2.4) are considered as a system in C_1 and C_2, then, in order that the homogeneous system should have a non-zero solution, it is necessary and sufficient that its determinant should not be equal to zero, i.e.

$$\Delta = \lambda^2 - 2\lambda\,[1 + g(\nu^2 - \nu - 1)] + \nu(1+\nu) g\,[(1-\nu)(2-\nu)g - 2] = 0 \tag{2.7}$$

It is curious that (2.7) is identical to the well-known Polubarinova-Kochina condition for the point G which is the end of the cut (/1/, p.255).

3. *Construction of the functions F and Z by the Polubarinova-Kochina method.* According to this method /1/, the functions F and Z are the solutions of a certain linear second-order differential equation of the Fuchs class with regular singular points which are the singular points of the functions ω and z such that

$$w = F/Z \tag{3.1}$$

By determining the indices of the functions F and Z around the singular points we find that these functions are linear combinations of the two branches of the following Riemann function:

$$\frac{1}{(\xi+\xi_A)\sqrt{\xi(1-\xi)^{1+\nu}}}\,\mathbf{P}\begin{Bmatrix} 0 & 1 & g & \infty & \\ 0 & 0 & 0 & -\dfrac{1+\nu}{2} & \xi \\ \dfrac{1}{2} & \nu & 2 & -\dfrac{\nu}{2} & \end{Bmatrix} = \frac{v}{(\xi+\xi_A)\sqrt{\xi(1-\xi)^{1+\nu}}} \tag{3.2}$$

where v is the solution of Eq.(2.1).

Using (3.2), (3.1) and (2.2), we find that

$$\frac{d\omega}{dt} = A\frac{f_1(t)}{h_-(t)},\quad \frac{dz}{dt} = \frac{A}{\gamma\rho}\frac{f_2(t)}{h_-(t)} \tag{3.3}$$

$$h_{\pm}(t) = \operatorname{ch} 2t \pm \cos 2a,\quad A > 0,\quad a = \operatorname{arctg}\sqrt{\xi_A}$$

It can be checked that the functions (1.3), which are defined on the basis of the relationships (3.3) and (2.2), satisfy the boundary conditions (1.1) which are written in terms of the above-mentioned functions and are therefore a parametric solution of the initial boundary value problem.

4. ***Second method. Finding the functions F and Z by the Fourier integral method.*** In the case under consideration the shape of the domain z also enables one to solve the problem using a Fourier integral. In order to satisfy the boundary condition $\operatorname{Im} F = 0$ on the line of separation, we put an equal flow rate symmetric point sink on the imaginary axis below this boundary and apply the so-called method of mirror images /1, 2, 7, 8/. We seek the functions F and Z in the half band of the plane t (Fig.2,b) in the form (here and subsequently, integration with respect to α is carried out from 0 to ∞)

$$\begin{aligned} dz/dt &= \int [C_1(\alpha)\cos\alpha t + iC_2(\alpha)\sin\alpha t]\,d\alpha \\ d\omega/dt &= (2\pi)^{-1}Q\,[1/(t-ia) + 1/(t+ia)] + \int C_3(\alpha)\sin\alpha t\,d\alpha \end{aligned} \tag{4.1}$$

where $C_i(\alpha)$ $(i = 1, 2, 3)$ are certain unknown functions which are to be determined. It is obvious that, for any $C_i(\alpha)$, the functions F and Z satisfy the boundary conditions (1.1) on the sides AB and AD and the second condition on the side BC which is written in terms of these functions.

In order to use the remaining three boundary conditions subsequently, we shall employ the following integral representations (/9/, p.491):

$$\frac{t}{t^2+a^2} = \begin{cases} \int e^{-a\alpha} \sin \alpha t \, d\alpha, & v < a \\ -i \int e^{i\alpha t} \operatorname{ch} \alpha a \, d\alpha, & v > a \end{cases}$$

We now take account of the fact that the straight line $v = 0$ corresponds to the line of separation BC in the t-plane while the line $v = \pi/2$ corresponds to the depression curve. By separating the real and imaginary parts in the expressions for F and Z and satisfying the boundary conditions on CD, which have not been used up to now, and the first condition on BC, we get a system of linear algebraic equations for finding the unknown functions $C_i(\alpha)$

$$\begin{gathered} \operatorname{sh} {}^1/_2\pi\alpha C_1(\alpha) - \operatorname{ch} {}^1/_2\pi\alpha\,[C_2(\alpha) + C_3(\alpha)] = \pi^{-1}Qe^{-{}^1/_2\pi\alpha} \operatorname{ch} \alpha a \times \\ \rho C_2(\alpha) - C_3(\alpha) = \pi^{-1}Qe^{-\alpha a} \\ \varepsilon \operatorname{ch} {}^1/_2\pi\alpha C_1(\alpha) - \operatorname{sh} {}^1/_2\pi\alpha\,[\varepsilon C_2(\alpha) - C_3(\alpha)] = \pi^{-1}Qe^{-{}^1/_2\pi\alpha} \operatorname{ch} \alpha a \end{gathered} \tag{4.2}$$

By solving system (4.2), the determinant of which can be shown to be non-zero, we find $C_i(\alpha)$. By introducing the resulting values of $C_i(\alpha)$ into Eqs.(4.1) and making use of the well-known (/9/, p.519) values of the integrals, we finally get

$$\begin{gathered} \frac{d\omega}{dt} = \frac{Q}{\pi h_-(t)}[B_1 \operatorname{sh}(1+\nu)t + B_2 \operatorname{sh}(1-\nu)t], \quad \frac{dz}{dt} = \frac{Q}{\pi\gamma\rho h_-(t)} \times \\ \{B_1[\operatorname{ch}(1+\nu)t + i\gamma \operatorname{sh}(1+\nu)t] - B_2[\operatorname{ch}(1-\nu)t - i\gamma \operatorname{sh}(1-\nu)t]\} \end{gathered} \tag{4.3}$$

$$\begin{gathered} (B_1 = \cos(1-\nu)\,a + \gamma \sin(1-\nu)\,a,\ B_2 = \cos(1+\nu)\,a - \\ - \gamma \sin(1+\nu)\,a) \end{gathered}$$

By taking account of the fact that $C = (B_1 + B_2)/(B_1 - B_2)$ and introducing the constant A which is connected with the flow rate Q by the relationship

$$Q = \pi A\,[2 \sin a\,(\sin a\nu + \gamma \cos a\nu)]^{-1} \tag{4.4}$$

we again arrive at (3.3).

5. ***Calculation of the lens. Discussion.*** Eqs.(3.3) constitute a parametric solution of the problem in the case of the source. Let us now extend the results which have been obtained to the case of an irrigator with a small cross-section which is close to being semicircular. In order to do this, we shall take, as the contour of the irrigator, the line of equal head which passes through the upper point of the cross-section of the irrigator M_0 with the coordinates $x = 0,\ y = {}^1/_2 D$, where D is the diameter of the irrigator and denote the affix of this point in the t-plane by μ (Fig.2,b). We shall assume that the head is equal to h_0 on the contour of the irrigator. Eqs.(3.3) then contain three unknown constants a, μ and A. The radius of the irrigator ${}^1/_2 D$, the head acting on its contour h_0 and the depth H_0 of the initial (up to the formation of the lens) surface of the saline ground water, for which we have the following equation /1/:

$$LH_0 = -\int_0^L y_{BC}(x)\,dx \tag{5.1}$$

serve to determine these constants.

By integrating (3.3) along the corresponding parts of the boundary of the domain t, we get

$$h_0 = A \int_\mu^{{}^1/_2\pi} [\cos\tau \sin\nu\tau + C \sin\tau \cos\nu\tau] \frac{d\tau}{h_-(\tau)} \tag{5.2}$$

$$\frac{1}{2} D = \frac{A}{\gamma\rho} \int_a^\mu [\cos\tau(\cos\nu\tau - \gamma \sin\nu\tau) - C \sin\tau(\sin\nu\tau + \gamma\cos\nu\tau)] \frac{d\tau}{h_-(\tau)}$$

The system of Eqs.(5.1) and (5.2) determine the required parameters. After they have been found, the magnitude of the flow rate Q is determined using formula (4.4), the width of the lens is determined using formula (1.2), and the thickness (power) of the lens at the maximum cross-section $T = H + H_1$, where H is the maximum height of the depression curve, H_1 is the greatest distance between the boundary of separation and the level at which the foundation of the irrigator is laid while H_2 is the smallest distance. Dimensionless quantities were used: all the linear characteristics (the dimensions of the lens L, H, H_1, H_2 and T, including the filtration flow rate) are relative to the magnitude of H_0, while the previous notation is retained for these quantities.

Let us now derive the parametric equations for the coordinates of the two required boundaries on the lens $(0 \leqslant \tau < +\infty)$:
in the case of the free surface CD

$$x = \frac{A \cos {}^1/_2 \pi\nu}{\varepsilon} \int_0^\tau [\operatorname{sh}\tau \operatorname{sh}\nu\tau + C \operatorname{ch}\tau \operatorname{ch}\nu\tau] \frac{d\tau}{h_+(\tau)}$$
$$y = H - A \sin {}^1/_2 \pi\nu \int_0^\tau [\operatorname{sh}\tau \operatorname{ch}\nu\tau + C \operatorname{ch}\tau \operatorname{sh}\nu\tau] \frac{d\tau}{h_+(\tau)} \tag{5.3}$$

in the case of the surface of separation BC

$$x = \frac{A}{\gamma\rho} \int_0^\tau [\operatorname{ch}\tau \operatorname{ch}\nu\tau + C \operatorname{sh}\tau \operatorname{sh}\nu\tau] \frac{d\tau}{h_-(\tau)} \tag{5.4}$$
$$y = -H_1 + \frac{A}{\rho} \int_0^\tau [\operatorname{ch}\tau \operatorname{sh}\nu\tau + C \operatorname{sh}\tau \operatorname{ch}\nu\tau] \frac{d\tau}{h_-(\tau)}$$

The depression curve and the line of separation calculated when $\varepsilon = 0.08$; $\rho = 0.3$; $H_0 = 0.5$; $D = 0.4$ and $h_0 = 0.5$ are shown in Fig.1. The results of calculations of the dimensions of the lens L and T and of the flow rate Q for certain values of the parameters ε, ρ, D and h_0 are shown in Tables 1 and 2. The tables consists of several sections in each of which one of the above-mentioned parameters is changed while the remaining parameters are fixed at the values of $\varepsilon = 0.01$; $\rho = 0.1$; $H_0 = 1.0$; $D = 0.3$ and $h_0 = 0.4$.

Table 1

ε	L	T	Q	ρ	L	T	Q
10^{-4}	8497	139.4	0.849	0.02	175.3	50.8	1.753
10^{-3}	1097	55.7	1.097	0.05	163.7	32.7	1.637
$5\cdot10^{-3}$	275	30.1	1.374	0.2	140.0	16.1	1.401

Table 2

D	L	T	Q	h_0	L	T	Q
0.05	86.6	13.1	0.866	0.1	68.7	10.4	0.687
0.1	105.5	15.9	1.055	0.4	152.4	23.0	1.524
0.3	152.4	23.0	1.524	0.6	197.8	29.9	1.978

The results in Table 1 enable one to draw certain conclusions regarding the effect of the parameters ε and ρ on the dimensions of the lens. It is seen that as ε becomes smaller, the lens increases in size, mainly by becoming wider, since the increase in L is proportional to $1/\varepsilon$ (see (1.2)). As ρ decreases, that is, there is a weakening of the support on the part of the saline waters, a considerable increase in the thickness of the lens is observed, that is, the lens now mainly becomes deeper. We note that, as far as the effect of the parameters ε and ρ is concerned, a certain similarity can be seen between the lens being discussed and the lenses in channels which have been described in /10/.

The calculations presented in Table 2, which study the effect of the diameter D and the magnitude of the head h_0 on the shape and dimensions of the lenses, show that there is a direct proportionality between the magnitude of L and T.

The behaviour of the point of inflection G, which is marked in Fig.1 with a small cross, is of particular interest. It is found that, as the parameter ρ is increased for fixed values of ε and H_0, the point of inflection G moves along the free surface (according to the results of Sect.2, $g > 1$ in this case) and subsequently passes onto the line of separation (then $g < 1$). In the example cited in Fig.1, the point of inflection G is located on the line of separation and has the coordinates $x = 2.128$ and $y = -0.961$.

The author thanks D.F., Shul'gin for advice and comments which helped to improve this paper.

REFERENCES

1. POLUBARINOVA-KOCHINA P.YA., Theory of the Motion of Ground Waters, Nauka, Moscow, 1977.
2. RIZENKAMPF B.K., On one case of water filtration in a multilayer soil, Uch. Zap. Saratov. Univ., Ser. Gidravliki, 15, 5, 1940.
3. SHUL'GIN D.F. and NOVOSEL'SKII S.N., Mathematical Models and Methods of calculating Moisture Transport during Intraground Irrigation, in: Mathematics and Problems of Water Utilization, Naukova Dumka, Kiev, 1986.
4. KOPPENFELS W. and STALLMANN F., Praxis der konformen Abbildung, Springer, Berlin, 1959.
5. CHIBRIKOVA L.I., On the use of the Riemann boundary value problem to construct integral representations of certain solutions of equations of the Fuchs class, in: Theory of Functions of a Complex Variable, Izd. Chuvash. Univ., Cheboksary, 1983.
6. TSITSKISHVILI A.R.,On the conformal transformation of a half plane onto curved quadrilaterals, Dokl. Akad. Nauk SSSR, 233, 4, 1977.
7. ARAVIN V.I. and NUMEROV S.N., Theory of the Motion of Liquids and Gases in an Undeformable Porous Medium, Gostekhizdat, Moscow, 1953.
8. POLUBARINOVA-KOCHINA P.YA.(Ed). Development of Investigations in Filtration Theory in the USSR, (1917-1967), Nauka, Moscow, 1969.
9. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series and Products, Nauka, Moscow, 1971.
10. EMIKH V.N., On the shape of a fresh water lens during filtration from a channel, Izv. Akad. Nauk SSSR, Mekh. Zhid. i Gazov, 2, 1966.

Translated by E.L.S.

PMM U.S.S.R.,Vol.53,No.3,pp.355-361,1989 0021-8928/89 $10.00+0.00
Printed in Great Britain

ESTIMATES OF THE FLOW RATE CHARACTERISTICS IN THE THEORY OF FILTRATION AND HEAT CONDUCTION*

M.M. ALIMOV and E.V. SKVORTSOV

In developing the approach proposed in /1, 2/, it is shown that it is possible to obtain estimates of the flow rate characeristics in the case of spatial, stationary linear filtration of an incompressible fluid in an inhomogeneous porous medium. The volume of the filtration domain and the area of a segment of a boundary of indeterminate form are employed as the decisive geometric characteristics (in the planar case, it is the area of the domain and the length of a segment of the boundary of indeterminate form). The corresponding boundary value problems are formulated. The subdomains of the domain of existence of a solution in which the extremal estimate is a lower estimate are indicated. An example is given.

**Prikl.Matem.Mekhan.*,53,3,462-468,1989